Appendix I

Let $C \in \mathcal{C}(N)$ be fixed, and let

$$\mathbf{x}^* = \text{argmin}_{\mathbf{x} \in \mathcal{R}(C)} H(\mathbf{x}),$$

be the optimal solution, and let $\hat{\mathbf{x}} = \hat{\mathbf{x}}(C)$ be the output of DD for C. For each $D \in \mathcal{D}$, denote $\hat{\mathbf{x}}_D = \hat{\mathbf{x}}_D(C)$. From the positivity of ϕ_{vw}, for all $D \in \mathcal{D}$,

$$\sum_{v \in V} \phi_v(x_v^*) + \sum_{(v,w) \in E_D} \phi_{vw}(x_v^*, x_w^*) \leq H(x^*). \tag{1}$$

From the minimality of $\hat{\mathbf{x}}_D$ in each connected components of D, for all $D \in \mathcal{D}$,

$$\begin{aligned} &\sum_{v \in V} \phi_v((\hat{\mathbf{x}}_D)_v) + \sum_{(v,w) \in E_D} \phi_{vw}((\hat{\mathbf{x}}_D)_v, (\hat{\mathbf{x}}_D)_w) \\ \leq &\sum_{v \in V} \phi_v(x_v^*) + \sum_{(v,w) \in E_D} \phi_{vw}(x_v^*, x_w^*). \end{aligned} \tag{2}$$

From (1), (2) and the definition of $\hat{\mathbf{x}}$,

$$\begin{aligned} &\sum_{D \in \mathcal{D}} \left[\sum_{v \in V} \phi_v(\hat{\mathbf{x}}_v) + \sum_{(v,w) \in E_D} \phi_{vw}(\hat{\mathbf{x}}_v, \hat{\mathbf{x}}_w) \right] \\ \leq &\sum_{D \in \mathcal{D}} \left[\sum_{v \in V} \phi_v((\hat{\mathbf{x}}_D)_v) + \sum_{(v,w) \in E_D} \phi_{vw}((\hat{\mathbf{x}}_D)_v, (\hat{\mathbf{x}}_D)_w) \right] \\ \leq &\sum_{D \in \mathcal{D}} \left[\sum_{v \in V} \phi_v(x_v^*) + \sum_{(v,w) \in E_D} \phi_{vw}(x_v^*, x_w^*) \right] \\ \leq &|\mathcal{D}| H(x^*). \end{aligned} \tag{3}$$

By the property (2) of Lemma 2, i.e. from the property that for each edge e of E, the number of decompositions in $\mathcal{D}$ that removes e is at most $\varepsilon|\mathcal{D}|$, we obtain that

$$\begin{aligned} &(1-\varepsilon)|\mathcal{D}|H(\hat{\mathbf{x}}) \\ \leq &\sum_{D \in \mathcal{D}} \left[\sum_{v \in V} \phi_v(\hat{\mathbf{x}}_v) + \sum_{(v,w) \in E_D} \phi_{vw}(\hat{\mathbf{x}}_v, \hat{\mathbf{x}}_w) \right] \end{aligned} \tag{4}$$

From (3) and (4), we have

$$(1-\varepsilon) H(\hat{\mathbf{x}}_D) \leq H(\mathbf{x}^*).$$

Appendix II

Computing g_i

- Let $V_i = \{(a,b) | a,b \in \{1,2,\dots \frac{1}{\varepsilon}\}\}$ be the set of vertices of R_i.
- Order the elements of V_i by dictionary order, i.e., $(a_1,b_1) < (a_2,b_2)$ if $a_1 < a_2$ or, $a_1 = a_2$ and $b_1 < b_2$. Let $v_1, v_2, \dots, v_{\frac{1}{\varepsilon^2}}$ be the vertices in that order.
- For $t = 0,1,\dots,\left(|V_i| - \frac{1}{\varepsilon}\right) = t^*$, let
  $B_t = \left\{v_{t+1},\dots,v_{t+\frac{1}{\varepsilon}}\right\}$.
  Let $V_{it} = \{v \in V_i \mid$ order of v is less than or equal to some vertex in $B_t\}$. Let E_{it} be the set of edges that connect two vertices of V_{it}.
- For each assignment $\hat{\mathbf{x}}^{B_t} \in [k]^{|B_t|}$ over B_t, and each $C_{(t)} = (C_{t1}, C_{t2}, \dots, C_{tk}) \in \mathcal{C}(|V_t|)$, let
  $$\mathcal{R}(\hat{\mathbf{x}}^{B_t}, C_{(t)}) = \mathcal{R}(C_{(t)}) \bigcap \left\{\mathbf{x} \in [k]^{|V_{it}|} \mid \mathbf{x}_v = \hat{\mathbf{x}}_v^{B_t}\ \forall\, v \in B_t\right\}.$$
  We will compute the following for $t = 0,1,\dots,t^*$).
  $$\hat{g}_t(\hat{\mathbf{x}}^{B_t}, C_{(t)}) = \min_{\mathbf{x} \in \mathcal{R}(\hat{\mathbf{x}}^{B}, C_{(t)})} \left[\sum_{v \in V_{it}} \phi_v(\mathbf{x}_v) + \sum_{(v,w) \in E_{it}} \phi_{vw}(\mathbf{x}_v, \mathbf{x}_w)\right].$$
- For $t = 0$, note that $V_{i0} = B_0$. Hence we directly compute $\hat{g}_0(\hat{\mathbf{x}}^{B_0}, C_{(0)})$ for all $\hat{\mathbf{x}}^{B_0} \in [k]^{|B_0|}$ and $C_{(0)} \in \mathcal{C}(|V_{i0}|)$.
- For $t = 1,2\dots t^*$,
  - For each t, let $B'_t = B_t \bigcup B_{t-1}$. For each $\hat{\mathbf{x}}^{B'_t} \in [k]^{|B'_t|}$, and $C_{(t)} \in \mathcal{C}(|V_t|)$ let
    $$\mathcal{R}(\hat{\mathbf{x}}^{B'_t}, C_{(t)}) = \mathcal{R}(C_{(t)}) \bigcap \left\{\mathbf{x} \in [k]^{|V_{it}|} \mid \mathbf{x}_v = \hat{\mathbf{x}}_v^{B'_t}\ \forall\, v \in B'_t\right\},$$
    and compute
    $$\hat{g}'_t(\hat{\mathbf{x}}^{B'_t}, C_{(t)}) = \min_{\mathbf{x} \in \mathcal{R}(\hat{\mathbf{x}}^{B'_t}, C_{(t)})} \left[\sum_{v \in V_{it}} \phi_v(\mathbf{x}_v) + \sum_{(v,w) \in E_{it}} \phi_{vw}(\mathbf{x}_v, \mathbf{x}_w)\right]$$
    by the relation
    $$\begin{aligned}\hat{g}'_t(\hat{\mathbf{x}}^{B'_t}, C_{(t)}) &= \hat{g}_{t-1}\left(\left(\hat{\mathbf{x}}^{B'_t}\right)_{B_{t-1}}, C'_{(t)}\right) \\ &+ \phi_{v_{t+\frac{1}{\varepsilon}}}\left(\left(\hat{\mathbf{x}}^{B'_t}\right)_{v_{t+\frac{1}{\varepsilon}}}\right) \\ &+ \phi_{v_{t+\frac{1}{\varepsilon}}, v_{t+\frac{1}{\varepsilon}-1}}\left(\left(\hat{\mathbf{x}}^{B'_t}\right)_{v_{t+\frac{1}{\varepsilon}}}, \left(\hat{\mathbf{x}}^{B'_t}\right)_{v_{t+\frac{1}{\varepsilon}-1}}\right) \\ &+ \phi_{v_{t+\frac{1}{\varepsilon}}, v_t}\left(\left(\hat{\mathbf{x}}^{B'_t}\right)_{v_{t+\frac{1}{\varepsilon}}}, \left(\hat{\mathbf{x}}^{B'_t}\right)_{v_t}\right),\end{aligned}$$

where $C'_{tj} = C_{tj} - 1$ for $j \in [k]$ such that $\left(\hat{\mathbf{x}}^{B'_t}\right)_{v_{t+\frac{1}{\varepsilon}}} = j$, and $C'_{tj} = C_{tj}$ for other j's. In the above computation, when there is no edge between $v_{t+\frac{1}{\varepsilon}}$ and $v_{t+\frac{1}{\varepsilon}-1}$, the term $\phi_{v_{t+\frac{1}{\varepsilon}}, v_{t+\frac{1}{\varepsilon}-1}}$ is not computed.

- For $\hat{\mathbf{x}}^{B_t} \in [k]^{|B_t|}$ and $C_{(t)} \in \mathcal{C}(|V_t|)$, compute

$$\hat{g}_t(\hat{\mathbf{x}}^{B_t}, C_{(t)}) = \min_{j \in [k]} \hat{g}'_t(\left(j, \hat{\mathbf{x}}^{B_t}\right), C_{(t)}).$$

– For each $C_{(i)} \in \mathcal{C}(|V_i|)$, output

$$g_i(C_{(i)}) = \min_{\hat{\mathbf{x}}^{B_{t^*}} \in [k]^{|B_{t^*}|}} \hat{g}_{t^*}(\hat{\mathbf{x}}^{B_{t^*}}, C_{(i)}).$$